Exact Stiffness Matrices for Piles in Nonhomogeneous Elastic Foundation

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Abstract

This paper presents an exact solution for the load – settlement relationship of axially loaded piles embedded in nonhomogeneous elastic foundation. The governing differential equation is reduced to modified Bessel equation of order *v*. The solution is represented by Bessel's functions of the first kind of order *v*. The stiffness coefficients are then derived from the exact solution. Numerical comparison with approximate solutions of special cases verify the accuracy and efficiency of the adopted method.

تحليل الركائز المستندة على أساس غير متجانس بطريقة الصلادة

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الخلاصة

يتضمن البحث إيجاد الحل المضبوط لدالة الحمل - الإزاحة للركائز المحملة محورياً ومغمورة فيغي متجانس. اجري تحوير للمعادلة التفاضلية التي تحكم تصرف الركيزة وتحويلها الى معادلة بسل المعدلة وتـم ايجاالمعادلـة الناتجة باستخدام دوال بسل من الدرجة الاولى. استخدمت هذه الحلول في اشتقاق معاملات الصلادة تحليلية بسيطة اعتمدت بعض المسائل لاستعراض كفاءة ودقة الطريقة المقترحة من خلال مقارنة النتائج مع طريقـة العناصـر المحددة.

Introduction

Various methods have been developed to predict the settlement in a single pile. Such methods may be classified into three broad categories: load – transfer method, methods based on the theory of elasticity, and numerical methods.

The load – transfer method, proposed by Coyle and Reese [1], utilizes soil data measured from field tests on instrumented piles and laboratory tests on model piles. It is assumed that the movement of the pile at any point is related only to the shear stresses at that point and is independent of the stress elsewhere on the pile. Thus no proper account is taken of the continuity of the soil mass.

Elastic based analysis have been employed by several investigators. In most of their approaches, the pile is divided into a number of uniformly loaded elements, and a solution is obtained by imposing compatibility between the displacement of the pile and the adjacent soil for each element of the pile [2].

Among the numerical methods, the finite element method (FEM) has been considered as prominent procedure that has been used successfully for solution of pile problem. It can provide realistic and satisfactory solutions for many problems involving coupling or interaction between soils and structure [3].

Miyahara and Ergatoudis [4] discussed the casses of partly buried pile in elastic foundation for which the soil modulus was assumed to be constant. However, it is usually not a reasonable assumption to take the foundation coefficient to be uniform with length along the pile. Most soils become stiffer with depth in some cases linearly, in other with some other power of distance [5]. Essa [6] introduced a new approach to deal with piles on variable Winkler foundation to obtain the stiffness coefficients. The proposed method is based on using an approximate model in simulating the tangential soil reaction. The stiffness coefficients are then derived from the exact solution of the resulting differential equation.

In this work, the exact solution for the load-settlement relationship of axially loaded piles embedded in non-homogeneous elastic foundation is presented. Based on the resulting displacement function, the stiffness coefficients are then derived and given in closed form.

Governing Differential Equation

The differential equation which governs the behaviour of skin friction piles subjected to axial compression is given by [7]:

$$\frac{d^2u(x)}{dx^2} - \frac{S}{EA}K(x)u(x) = 0 \qquad (1)$$

in which;

u(x): displacement function of the element (m).

K(x): modulus of soil reaction function (N/m^3).

E: modulus of elasticity of the pile (N/m^2).

A: cross – sectional area of the pile (m^2).

S: pile perimeter (m).

Soil – Structure Interaction Model

The elastic properties of the soil may be constant with depth or varies linearly or nonlinearly depending on the type of soil. The moduli of subgrade reactions, as will be shown later, are functions of these properties. Therefore it is important to take this variation into account in the analysis of soil – pile interaction.

There are several distributions of the moduli of subgrade reactions along the pile length employed. The most widely used being that developed by Palmer and Thompson [8], which is of the form:

$$K(x) = k\left(\frac{x}{L}\right)^m \qquad (2)$$

where:

k: modulus of subgrade reaction (N/m^3)

L: pile length (m)

m: an empirical index equal to or greater than zero.

There are several assumptions about the patterns by which the soil moduli are varied, and then the corresponding values of the index (m). One of the common assumptions is that m= 0, for clay that is, the modulus is constant with depth [7]. Davisson and Parakash [9] suggested, however, that m= 0.15 is more realistic value for clays, in order to take into account the effect of some allowance for plastic soil behaviour at the surface.

In sand, however, Scott [10] considered that the soil modulus increase with depth, possibly linearly, that is the index m= 1, or in the form of square root, that is m= 0.5. These patterns of variation of the soil moduli with depth, which are shown in Fig. (1), are adopted in this study.

Stiffness Matrix Derivation

The differential equation for the displacement of pile supported on a variable elastic foundation may be produced by substituting eq. (2) into eq. (1) to get:

$$\frac{d^2u}{dx^2} - \lambda^2 x^m u = 0 \qquad (3)$$

where:

$$\lambda^2 \frac{Sk}{EAL^m} \qquad (4)$$

Changing the dependent variable u to y by means of the substitution:

$$u = yx^{\frac{1}{2}} \qquad (5)$$

the differential eq. (3) reduced to:

$$\frac{d^2y}{dx^2} - x^{-1}\frac{dy}{dx} - \left(\frac{x^{-2}}{4} + \lambda^2 x^m\right)y = 0 \qquad (6)$$

and changing the independent variable x to z by means of the substitution:

$$z = \left(\frac{2}{m+2}\right)\lambda x^{\left(\frac{m+2}{2}\right)} \qquad (7)$$

eq. (6) becomes an equation in y and z:

$$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} - (z^2 + \nu^2)y = 0 \qquad (8)$$

where:

$$\nu = \frac{1}{m+2} \tag{9}$$

Equation (8) is called the modified Bessel equation of order ν [11]. The solutions of eq. (8) are given by:

$$y_1(z) = I_\nu(z) \tag{10-a}$$

$$y_2(z) = I_{-\nu}(z) \tag{10-b}$$

where the functions $I_\nu(z)$ and $I_{-\nu}(z)$ are called the modified Bessel functions of the first kind of order ν and they have the representations [11]:

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{z^{2n+\nu}}{2^{2n+\nu} n!\Gamma(n+\nu+1)} \tag{11-a}$$

$$I_{-\nu}(z) = \sum_{n=0}^{\infty} \frac{z^{2n-\nu}}{2^{2n-\nu} n!\Gamma(n-\nu+1)} \tag{11-b}$$

Performing back substitutions for the changed variables used in eqs. (5 and 7), the solution equations (eq. (10)) reduced to the form:

$$u_1(x) = \sqrt{x} I_\nu(2\nu\lambda X^{\frac{1}{2\nu}}) \tag{12-a}$$

$$u_2(x) = \sqrt{x} I_{-\nu}(2\nu\lambda X^{\frac{1}{2\nu}}) \tag{12-b}$$

Referring to eq.(9), the values of ν are always real numbers (for $m \geq 0$) and may not be integers. Consequently, the solutions given in eq. (12) are independent solutions and the general solution of eq. (3) may be written as:

$$u(x) = C_1\sqrt{x} I_\nu(2\nu\lambda X^{\frac{1}{2\nu}}) + C_2\sqrt{x} I_{-\nu}(2\nu\lambda X^{\frac{1}{2\nu}}) \tag{13}$$

where C_1 and C_2 are arbitrary constants.

Boundary Conditions

The boundary conditions are assumed to be of displacement type because of the intended application of the result in developing a displacement based stiffness matrix. At x=0, we have:

$u(0)= u_1$ and so

$$C_2 = (\nu\lambda)^\nu \Gamma(1-\nu) u_1 \tag{14-a}$$

and for x=L, we get:

$u(L)= u_2$ and so

$$\left[C_1 = \frac{u_2 - (\nu\lambda)^\nu \Gamma(1-\nu)\sqrt{L} I_{-\nu}(2\nu\lambda L^{\frac{1}{2\nu}}) u_1}{\sqrt{L} I_\nu(2\nu\lambda L^{\frac{1}{2\nu}})} \right] \tag{14-b}$$

Stiffness Matrix

The terms of the stiffness matrix, are defined as the holding actions at the ends of the element, due to a unit displacement. Accordingly, the terms in the stiffness matrix are:

$$[K] = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \tag{15}$$

where:

$$k_{11} = EA\left\{ (\nu\lambda)^{2\nu} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \frac{I_{-\nu}(2\nu\lambda L^{\frac{1}{2\nu}})}{(2\nu\lambda L^{\frac{1}{2\nu}})} \right\} \tag{16-a}$$

$$k_{12} = -EA\left\{ \frac{(\nu\lambda)^\nu}{\sqrt{L}\Gamma(1+\nu)} \frac{1}{I_\nu(2\nu\lambda L^{\frac{1}{2\nu}})} \right\} \tag{16-b}$$

$$k_{21} = k_{12} \tag{16-c}$$

$$k_{22} = EA\left\{\lambda L^{\frac{1-2\nu}{2\nu}} \frac{I_{\nu-1}(2\nu\lambda L^{\frac{1}{2\nu}})}{I_{\nu}(2\nu\lambda L^{\frac{1}{2\nu}})}\right\} \quad (16\text{-d})$$

Stiffness Coefficients for Constant Soil Reaction

For the case of ***constant*** soil reaction (K: constant), we have m= 0, $\nu = \frac{1}{2}$, and $\lambda^2 = \left(\frac{Sk}{EA}\right)$ For this case we can proof that:

$$\lim_{\nu \to 0.5}[K] = \frac{EA\lambda}{\sinh(\lambda L)}\begin{bmatrix}\cosh(\lambda L) & -1 \\ -1 & \cosh(\lambda L)\end{bmatrix} \quad (17)$$

Numerical Examples

To demonstrate the efficiency and accuracy of the method presented, two numerical examples are solved and compared with the finite element method (see Appendix).

Example 1

The analysis of a single pile of length (L) and square cross- section (0.4× 0.4m) as shown in Fig. (2) is considered. The surrounding soil is assumed as sandy soil with a value of (m= 1) or (m= 0.5). The main variables considered in this problem are the tangential modulus of subgrade reaction and the pile length. The end bearing resistance of the soil is nglected. For sandy soil the modulus of subgrade reaction is assumed to be ranged from k =5000 kN/m^3 (loose sand) to k= 15000 kN/m^3 (dense sand). The variation of pile ends displacements with soil modulus is shown in Fig. (3) for two cases m= 1 and m= 0.5. The length of the pile considered in this case is L= 20m. The effect of L/b (pile length/pile width) ratio on the pile ends displacement is shown in Fig. (4) for two cases m= 1 and m=0.5. The modulus of subgrade reaction adopted in this case is k= 15000 kN/m^3.

It is shown from Figs. (3-a, 4-a) that, when the value of m= 1, the results of the F.E.M. will be under estimated for different values of soil modulus k or L/b ratios. When the value of m= 0.5, the F.E.M. gives an over estimated displacement at the pile top and under estimated displacement at pile tip as shown in Fig. (3-b, 4-b).

Example 2

The same parameters which are used in example (1) are adopted in this case except the values of soil modulus and its variation along the pile. The surrounding soil is assumed to be clayey soil with value of (m= 0.15) or (m= 0). The tangential modulus of subgrade reaction is taken to be ranged from k= 2000 kN/m^3 (soft clay) to k= 8000 kN/m^3 (hard clay). The variation of pile ends displacement with soil modulus is shown in Fig. (5) for two cases m= 0.15 and m= 0. The length of the pile is taken to be L= 20m. The effect of L/b ratio on the pile ends displacements is shown in Fig. (6) for two cases m= 0.15 and m= 0. The modulus of subgrade reaction adopted in this case is k= 8000 kN/m^3.

Figs. (5 and 6) indicate that the results of F.E.M. will be overestimated for both the pile top and pile tip.

Conclusions

In this paper, an exact solution for the load – settlement relationship of axially loaded piles embedded in nonhomogeneous elastic foundation is performed. The stiffness coefficients of the pile element are derived directly and presented in closed form. Numerical examples shows a high efficiency of the adopted model.

Appendix

The stiffness coefficients for the axial element embedded in non-homogeneous elastic foundation can be expressed as follows:

$$[K]=[K]_a+[K]_f$$

where:

$[K]$: Element stiffness matrix.

$[K]_a$: Axial element stiffness matrix without elastic foundation.

$[K]_f$: Stiffness matrix of elastic foundation contribution.

The axial element stiffness matrix without elastic foundation given by [12]:

$$[K]_a=\frac{EA}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

If the soil modulus is represented by:

$$K(x)=k\left(\frac{x}{L}\right)^m$$

The finite element formulation of the elastic foundation stiffness matrix yield:

$$[K]_f=k\begin{bmatrix}\left(\frac{L}{m+1}-\frac{2L}{2m+1}+\frac{L}{3m+1}\right) & \left(\frac{L}{2m+1}-\frac{L}{3m+1}\right)\\ \left(\frac{L}{2m+1}-\frac{L}{3m+1}\right) & \left(\frac{L}{3m+1}\right)\end{bmatrix}$$

Notations

A: cross – sectional area of the pile (m^2).

C_1, C_2: constants defined in eq. (14).

E: modulus of elasticity of the pile (N/m^2).

I_ν, $I_{-\nu}$: modified Bessel functions of the first kind of order ν.

k: value of the soil modulus at the pile tip (N/m^3).

K(x): soil reaction function (N/m).

[K]: element stiffness matrix.

$[K]_a$: axial element stiffness matrix without elastic foundation.

$[K]_f$: stiffness matrix of elastic foundation.

L: length of the element.

m: an empirical index equal to or greater than zero.

S: pile perimeter (m).

u_1, u_2: nodal displacements at ends of the element.

u(x): displacement function of the element (m).

x: axial coordinate along the element.

y: variable defined in eq. (5).

z: variable defined in eq. (7).

λ : constant defined in eq. (4).

ν : constant defined in eq. (9).

Γ : gamma function.

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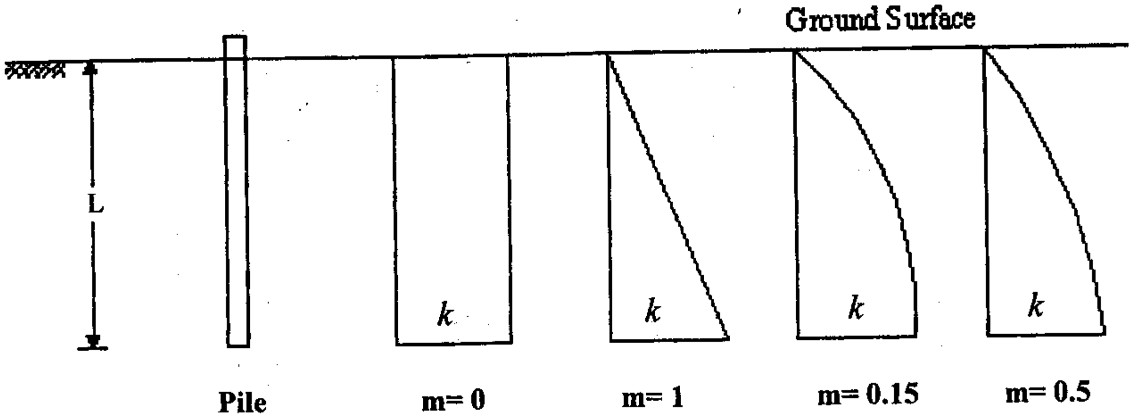

Fig. (1) Patterns of Soil Modulus Variation Along Pile Length

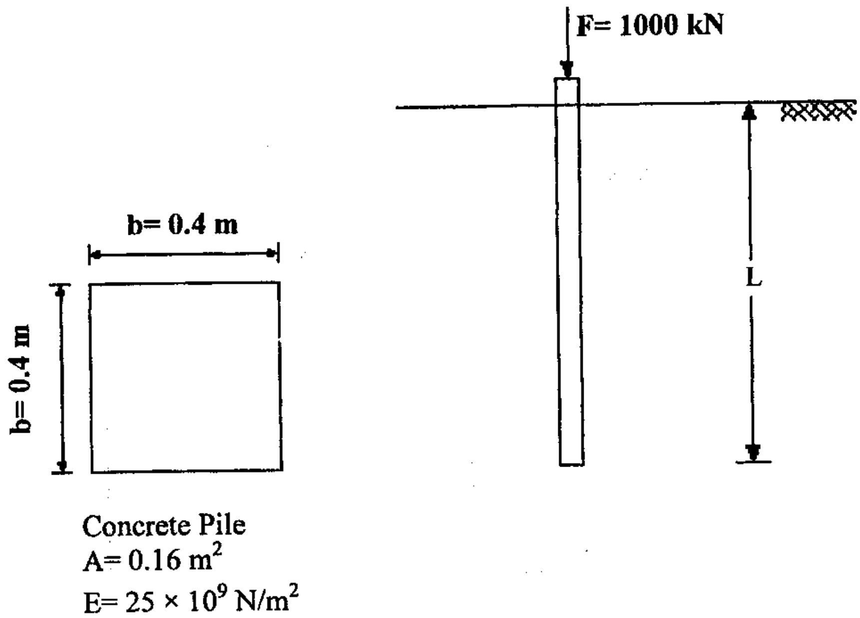

Fig. (2): Properties of Pile Used in Examples

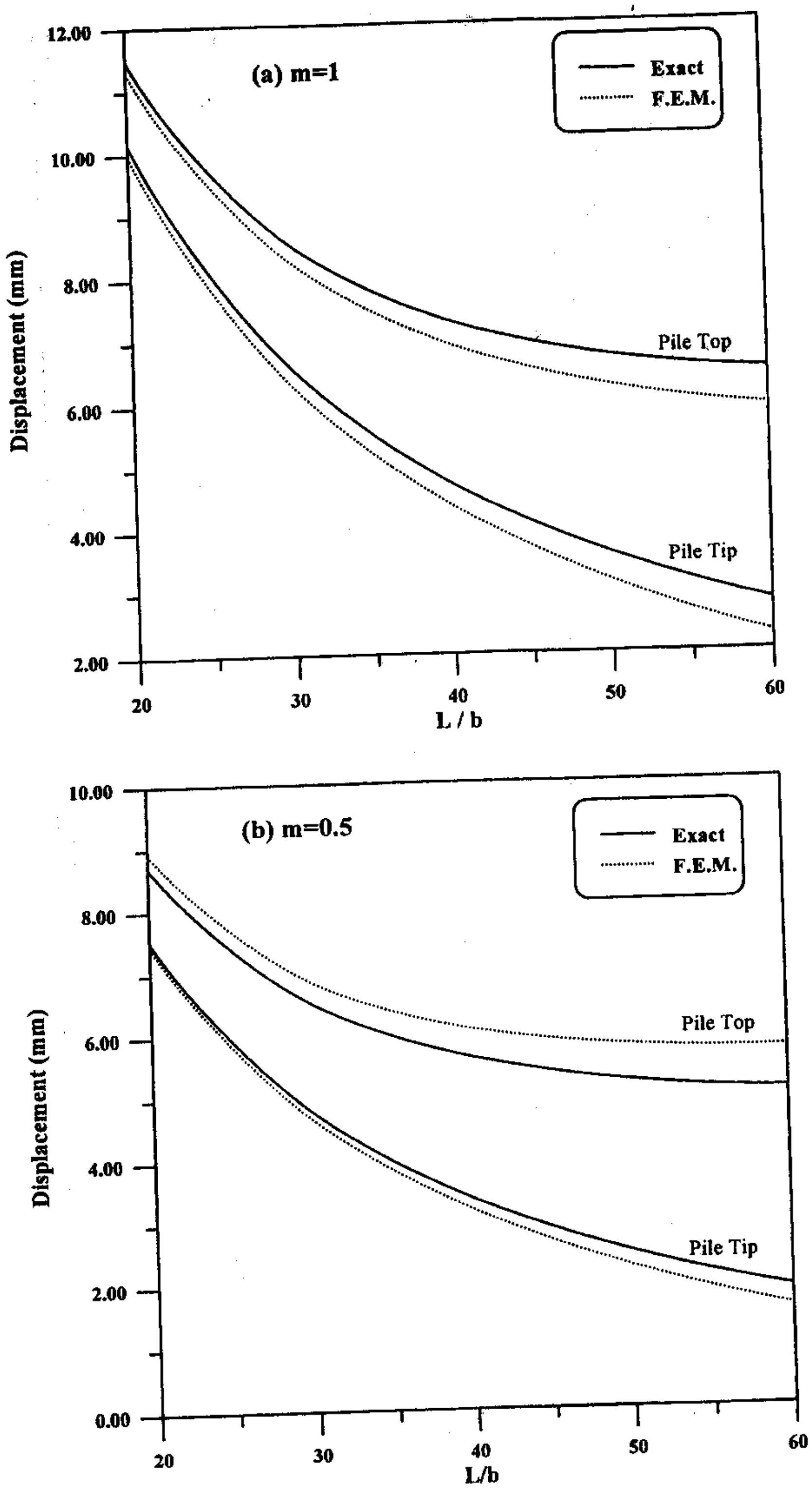

Fig. (4): Variation of Pile Ends Displacements with L/b ratio (Sandy Soil)

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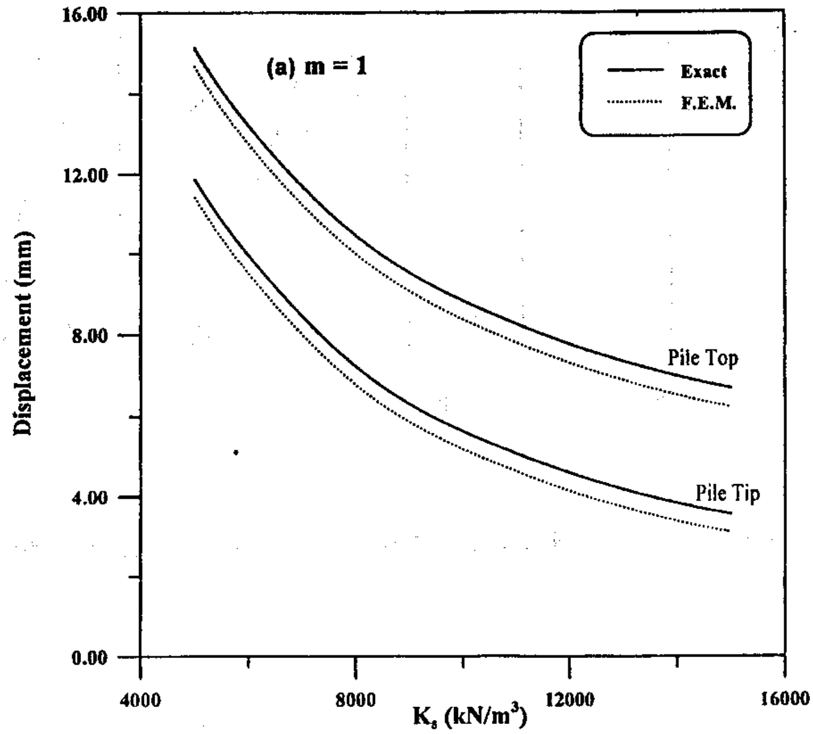

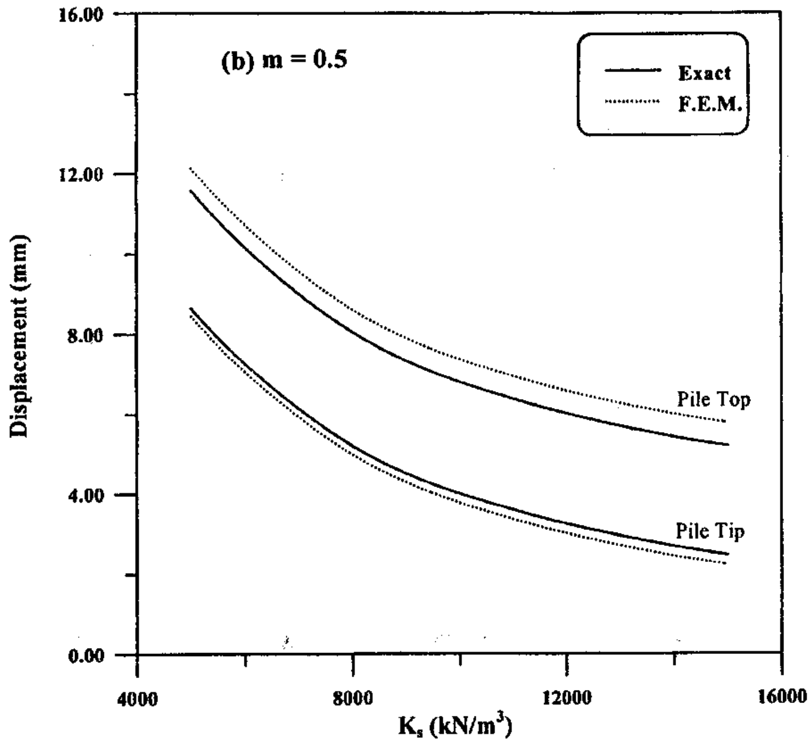

Fig. (3): Variation of Pile Ends Displacements with Soil Modulus (Sandy Soil)

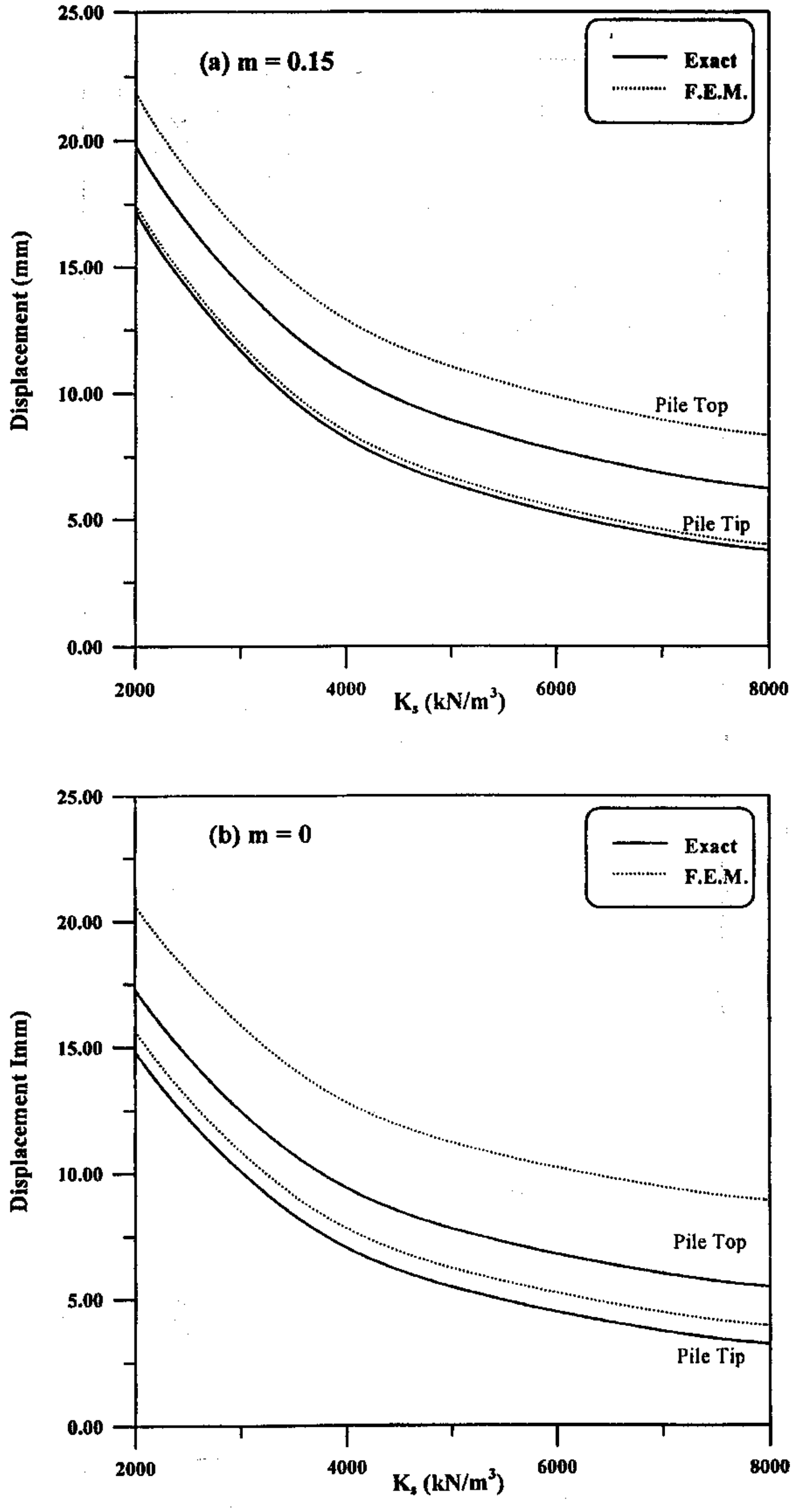

Fig. (5): Variation of Pile Ends Displacements with Soil Modulus (Clayey Soil)

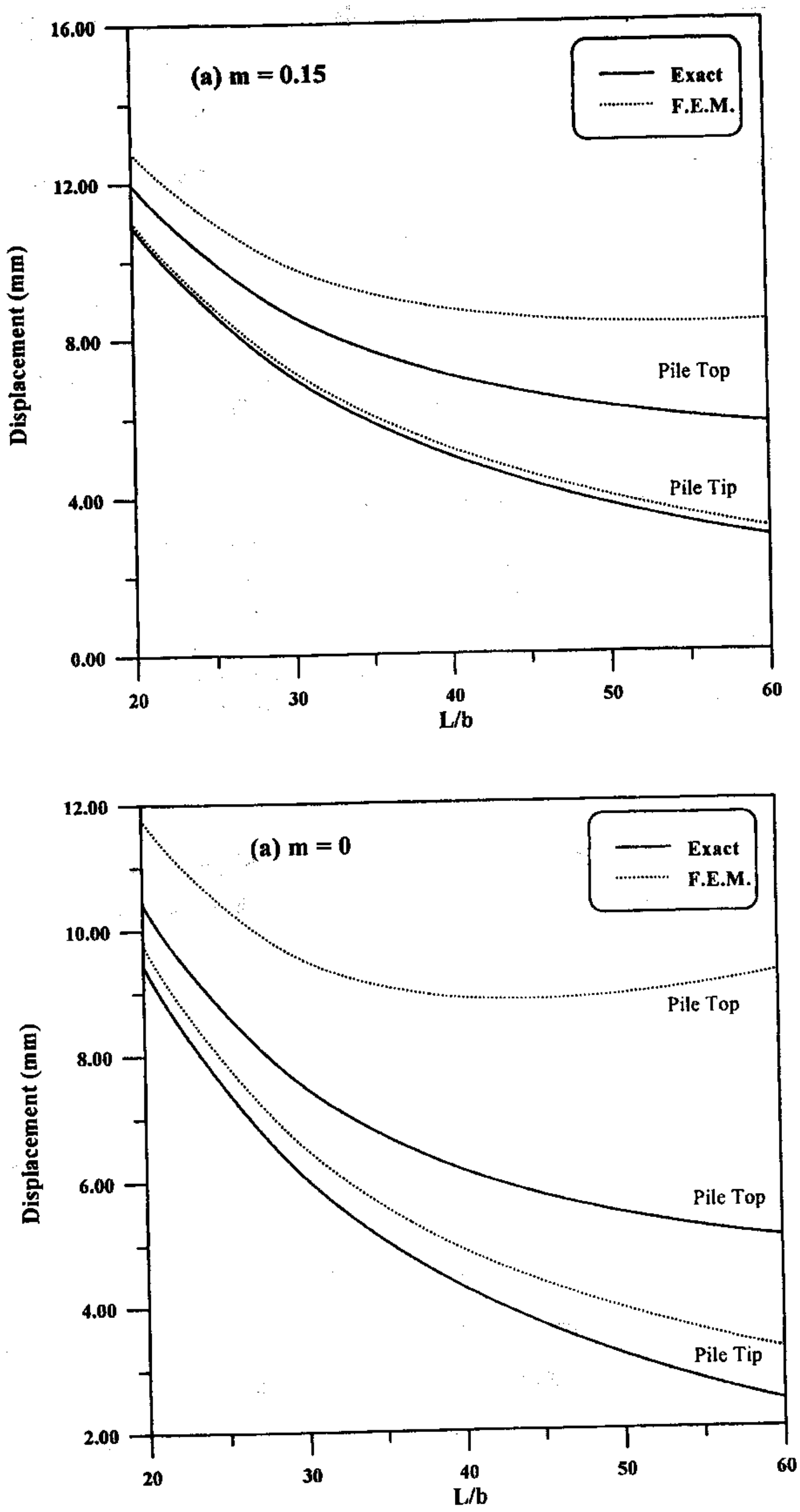

Fig. (6): Variation of Pile Ends Displacements with L/b ratio (Clayey Soil)